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Existence and uniqueness of growth cycles in post Keynesian systems

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Abstract

In this paper, we present a post Keynesian system in which investment is determined by the expected rate of profit and explore the existence and *uniqueness* of a periodic orbit (or a growth cycle) in it. We find that a periodic orbit appears if the marginal effect of the expected rate of profit on investment is strong enough and that it is unique if the speed of revision of expectation is fast enough.

Keywords: Keynesian economics; Expected rate of profit; Growth cycle; Uniqueness

JEL classification: E12; E32

1 Introduction

It is variations in investment that are viewed as the primary source of business cycles in the Keynesian theory. After the publication of Keynes' *General Theory*, the foundations of the Keynesian theory of business cycles were established by several economists such as Kalecki (1935, 1937), Harrod (1936), Samuelson (1939), Kaldor (1940), Metzler (1941), Hicks (1950) and Goodwin (1951).[1] Theories and models proposed by them have a common characteristic in that they stand upon the Keynesian principle of effective demand (including income or quantity adjustment induced by it) and that they emphasize variations in investment as a main factor for business cycles, but they differ in what determines investment expenditure. Indeed, Kalecki (1935, 1937) and Kaldor (1940) postulated that investment demand is determined by the rate of profit, which is related positively to aggregate income and negatively to capital stock (and negatively to the rate of interest), while Harrod (1936), Samuelson (1939), Metlzer (1941), Hicks (1950) and Goodwin (1951) that investment (including inventory investment) is related positively and directly to a change in aggregate income. Both of these postulates can be justified from theoretical or empirical viewpoints, but they lack some important aspect in Keynes' (1936, chap. 11) original theory of investment. In the

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[1]Precisely speaking, Kalecki (1935) and Harrod (1936) preceded Keynes' (1936) *General Theory*, but their theories can be classified as "Keynesian" because they are based upon the Keynesian principle of effective demand.

above typical Keynesian theories and models of business cycles, the role of *expectation* is overlooked or slighted by employing the *current* level of income or the *current* rate of profit as a proxy of the *expected* level of demand (or income) or of the *expected* rate of profit, respectively.[2] According to Keynes' *General Theory*, however, investment is affected positively and strongly by what he called the marginal efficiency of capital, which has the same meaning as the *expected* prospected rate of profit on capital (and negatively by the rate of interest). Also, he maintained that business cycles are induced mainly by violent changes in the marginal efficiency of capital due to frequent revisions of the long-term expectation. For these reasons, it may be worthwhile to take explicit account of the role of expectation in the theory of business cycles.

The purpose of the present paper is to analyze the effect of expectation, represented by the expected rate of profit, on the occurrence (existence) and *uniqueness* of business cycles. In Section 2, we shall present a post Keynesian system of growth cycles which can describe not only business cycles but also economic growth. In particular, we shall formalize the investment (or capital formation) function by relating the rate of capital formation to the expected rate of profit. In Section 3, we shall examine the properties of our post Keynesian system and discuss the existence of a periodic orbit, which can be interpreted as a growth cycle, and the uniqueness of it. We shall clarify that a periodic orbit does appear if the (marginal) effect of the expected rate of profit on capital formation is sufficiently strong and that this periodic orbit is unique if the speed of revision of expectation is fast enough. In Section 4, we shall summarize our analysis and conclude this paper. In Appendix, we shall provide some mathematical theorem utilized in our analysis.

2 The post Keynesian system

We shall formalize a post Keynesian system of growth cycles, which is composed of differential equations of the expected rate of profit and of per-capita capital stock. To construct this system, we shall first give the consumption-saving and investment functions.

2.1 Consumption and savings

We shall, following the Keynesian tradition, represent aggregate consumption by the following functional form:

$$C = c_0 N + cY, \tag{1}$$

where c and c_0 are positive constants with $c < 1$. In (1), Y, C and N stand for aggregate income (or aggregate output), aggregate consumption and the size of population, respectively; c is the marginal propensity to consume and c_0 represents the base or fundamental level of individual consumption. The consumption function given by (1)

[2] Exceptionally, Benassy (1984) put forward a Keynesian (or non-Walrasian) model of business cycles by placing much stress upon the role of expected demand on investment.

can deal with the context of economic growth because aggregate base or fundamental consumption is proportionate to population.

We can thus define the saving function in the following form:

$$S = Y - C = sY - c_0 N, \tag{2}$$

where $s = 1 - c \in (0, 1)$. Needless to say, s is the marginal propensity to save.[3]

2.2 Investment

We shall formulate the investment function reflecting Keynes' (1936, chap. 11) theory of investment. Specifically, the investment function is presented in the following form:

$$I = f(r^e)K. \tag{3}$$

In (3), I, K, r^e stand for aggregate gross investment, aggregate capital stock and the expected rate of profit, respectively; f is the gross capital formation function. Equation (3) means that the rate of gross capital formation I/K is affected by the expected rate of profit r^e (and by the rate of interest, which is assumed to be constant in our analysis).[4] Since Keynes (1936, chap. 11) defined the marginal efficiency of capital as the expected rate of profit on capital, the investment function (3) can be viewed as consistent with his theory of investment.[5]

Concerning the capital formation function f, we shall make the following reasonable assumption.

Assumption 1. *The real-valued function $f : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable for every $r^e \in \mathbb{R}_+$, and the following condition is satisfied for every $r^e \in \mathbb{R}_+$:*

$$f'(r^e) > 0. \tag{4}$$

Condition (4) simply means that the capital formation function f is strictly increasing in the expected rate of profit r^e. In this respect, Assumption 1 is a natural one.

[3] Following Serrano (1995), Allain (2015) and Lavoie (2016) proposed the saving function in the following form:

$$S = s_p \pi Y - Z,$$

where s_p and π stand for the rate of savings of capitalists and the capital share (note that workers are assumed to spend their whole income and to save nothing); Z represents the autonomous component of aggregate demand expenditure of capitalists, which is assumed to change at a constant rate. In Allain (2015), Z was interpreted as government expenditure, while in Lavoie (2016), as capitalists' autonomous consumption. This saving function is identical with ours (2) if $s = s_p \pi$ and $c_0 N = Z$. Note that it is not explicitly assumed in our analysis that there are classes with different saving behavior. For implications of autonomous expenditure in Kaleckian models, see also Skott (2016).

[4] It is possible to provide microeconomic foundation for the capital formation function given in (3). For details, see Murakami (2016).

[5] Our investment (or capital formation) function has a lot in common with Benassy's (1984) one because he related the level of investment to the expected level of (aggregate) demand. However, his analysis of business cycles is different from ours in that he focused on short term economic fluctuations, in which the level of capital stock remains the same over time, while we allow for changes in the level of capital stock (and the size of population).

2.3 Equilibrium in the good-service market

We shall derive the output-capital ratio, which can also be regarded as the rate of utilization, from the Keynesian principle of effective demand. The good-service market equilibrium condition, or the IS balance condition, implies that aggregate income per unit of capital stock is determined by the following:

$$S = sY - c_0N = f(r^e)K = I,$$

or

$$u = \frac{1}{s}\Big(\frac{c_0}{k} + f(r^e)\Big). \tag{5}$$

In (5), u and k stand for the output-capital ratio (or the rate of utilization) Y/K and per capita capital stock K/N, respectively.

On the basis of the output-capital ratio u, defined in (5), we shall derive the (actual) rate of profit (on capital). For this purpose, we shall assume that the capital share or the profit share in aggregate income, denoted by $\pi \in (0, 1)$, is constant.[6] Since aggregate profit is equal to aggregate income multiplied by the capital share, the rate of profit can be derived as follows:

$$r = \frac{\pi Y}{K} = \pi u = \frac{\pi}{s}\Big(\frac{c_0}{k} + f(r^e)\Big). \tag{6}$$

In (6), r stands for the rate of profit. Note that the actual rate of profit r is determined by the expected rate of profit r^e through investment.

2.4 Capital formation

We shall describe the process of capital formation (or of capital accumulation) through investment. Since capital stock K is varied by investment (net of capital depreciation), this process can be represented by the following equation:

$$\dot{K}\Big(= \frac{dK}{dt}\Big) = I - \delta K = [f(r^e) - \delta]K, \tag{7}$$

where δ is a positive constant. In (7), δ stands for the constant rate of capital depreciation.

[6] According to Kalecki (1939), the capital share π is determined by the degree of monopoly. In this respect, it suffices for our assumption to suppose that the degree of monopoly is constant.

2.5 Revision of expectation

We shall formalize the revision process of the expected rate of profit. The expected rate of profit r^e is held by firms and not necessarily equal to the actual one r, and the former is (continually) revised on the basis of the latter. To describe this revision process, we shall assume that r^e is changed in response to the difference from r in the following way:

$$\dot{r}^e\Big(=\frac{dr^e}{dt}\Big)=\alpha(r-r^e)=\alpha\Big[\frac{\pi}{s}\Big(\frac{c_0}{k}+f(r^e)\Big)-r^e\Big], \tag{8}$$

where α is a positive constant. Equation (8) means that the expected rate of profit r^e is adaptively revised, and α can be interpreted as the speed of revision of expectation.

2.6 Population change

To deal with economic growth, we shall take account of changes in population. To this end, we shall suppose that the size of population is varied at a constant rate as follows:

$$\dot{N}\Big(=\frac{dN}{dt}\Big)=nN, \tag{9}$$

where n is a real constant. In (9), n is noting but the rate of population change, which can be zero or negative.

2.7 Full system: System (PK)

We can now summarize our equations (7), (8) and (9):

$$\dot{K}=[f(r^e)-\delta]K, \tag{7}$$

$$\dot{r}^e=\alpha\Big[\frac{\pi}{s}\Big(\frac{c_0}{k}+f(r^e)\Big)-r^e\Big], \tag{8}$$

$$\dot{N}=nN, \tag{9}$$

To discuss the possibility of growth cycles, we shall consider the dynamics (differential equation) of per capita capital stock k. The differential equation of k can easily be derived from (7) and (9) as follows:

$$\dot{k}\Big(=\frac{dk}{dt}\Big)=\frac{\dot{K}}{N}-\frac{K}{N}\frac{\dot{N}}{N}=[f(r^e)-(\delta+n)]k. \tag{10}$$

Therefore, we can complete our post Keynesian system in the following form:

$$\dot{r}^e = \alpha\Big[\frac{\pi}{s}\Big(\frac{c_0}{k} + f(r^e)\Big) - r^e\Big], \tag{8}$$

$$\dot{k} = [f(r^e) - (\delta + n)]k. \tag{10}$$

In what follows, we shall denote the system of equations (8) and (10) by "System (PK)" (to signify "Post Keynesian"). Note that per capita income is determined along every solution path of System (PK) through (5) (and $Y/N = u/k$).

3 Analysis

We shall proceed to analyze our System (PK) and examine the possibility of occurrence of a limit cycle, which can be interpreted as a persistent growth cycle.

3.1 Existence and uniqueness of equilibrium

We shall first define an equilibrium point (or a steady state) of System (PK). In our analysis, an equilibrium point of System (PK) is defined as a point $(r^e, k) \in \mathbb{R}^2_{++}$ at which we have $\dot{r}^e = \dot{k} = 0$.[7] Then, an equilibrium point of System (PK), (r^*, k^*), is derived as a solution of the following simultaneous equations.

$$0 = \frac{\pi}{s}\Big(\frac{c_0}{k} + f(r^e)\Big) - r^e,$$

$$0 = f(r^e) - (\delta + n).$$

Thus, the unique equilibrium point of System (PK), (r^*, k^*), is, provided that it exists, given by the following:

$$(r^*, k^*) = \Big(f^{-1}(\delta + n), \frac{\pi c_0}{sf^{-1}(\delta + n) - \pi(\delta + n)}\Big). \tag{11}$$

To ensure the existence (and uniqueness) of an equilibrium point of System (PK), we shall also impose the following assumption.

Assumption 2. *The following conditions are satisfied:*

$$f(0) < \delta + n < \lim_{r^e \to \infty} f(r^e), \tag{12}$$

$$f\Big(\frac{\pi(\delta + n)}{s}\Big) < \delta + n. \tag{13}$$

[7] Note that $k = 0$ is ruled out as an equilibrium value of k by this definition.

Under Assumption 1, condition (12) implies the existence and uniqueness of $f^{-1}(\delta+n) > 0$ (by the implicit function theorem). Also, we can see from (13) that, under Assumption 1, we have

$$\frac{\pi(\delta+n)}{s} < f^{-1}(\delta+n),$$

or

$$sf^{-1}(\delta+n) - \pi(\delta+n) > 0. \tag{14}$$

Then, the equilibrium point of System (PK), defined in (11), can be shown to exist on $\mathbb{R}^2_{++}$.

3.2 The positivity constraint

Before exploring the existence and uniqueness of a limit cycle, we shall make sure that every solution path of System (PK) with an initial condition $(r^e(0), k(0)) \in \mathbb{R}^2_{++}$ at $t = 0$ stays on $\mathbb{R}^2_{++}$ for all $t \geq 0$. By so doing, we can ensure that a periodic orbit, if it exists, lies entirely on $\mathbb{R}^2_{++}$, which is an economically meaningful domain. Since it is seen that $k(t) > 0$ for all $t \geq 0$ along each solution path with $(r^e(0), k(0)) \in \mathbb{R}^2_{++}$, it suffices for our purpose to ensure that $r^e(t) > 0$ for all $t \geq 0$ along each of such solution paths.

It follows from the continuity of solution paths of System (PK) (by Assumption 1) that if $\dot{r}^e > 0$ at $r^e = 0$ for every $k > 0$, every solution path cannot leave the domain of $\mathbb{R}^2_{++}$, provided that it starts on (or enters) this domain. Then, it is sufficient to assure that for every $k > 0$

$$\dot{r}^e|_{r^e=0} = \alpha\left[\frac{\pi}{s}\left(\frac{c_0}{k} + f(0)\right)\right] > 0. \tag{15}$$

For this condition to hold, we shall introduce the following assumption.

Assumption 3. *The following condition is satisfied:*

$$f(0) \geq 0. \tag{16}$$

Combined with Assumption 1, condition (16) implies that the (gross) capital formation function takes on a nonnegative value. Assumption 3 can thus be viewed as reasonable in that the rate of gross capital formation is nonnegative by definition. Note that condition (15) is fulfilled for every $k > 0$ under this assumption.

3.3 System (PK) reformulated

We shall reformulate System (PK) as a generalized Liénard system for the sake of our analysis.[8] By so doing, we can easily discuss the existence and uniqueness of a limit cycle in System (PK).

For reformulation, we shall introduce the following new variables:

$$x = r^e - r^*, \tag{17}$$

$$y = \ln k^* - \ln k, \tag{18}$$

where (r^*, k^*) is defined by (11) and ln represents the natural logarithm. Since we have shown that k^* is positive under Assumptions 1 and 2 and that k is always positive along every solution path of System (PK) with $(r^e(0), k(0)) \in \mathbb{R}^2_{++}$, we can take the natural logarithms of k^* and of k.

We shall substitute the variables defined in (17) and (18) in System (PK) to obtain the following system:

$$\dot{x} = \phi(y) - F(x), \tag{19}$$

$$\dot{y} = -g(x), \tag{20}$$

where

$$\begin{aligned} g(x) &= f(r^* + x) - (\delta + n) \\ &= f(f^{-1}(\delta + n) + x) - (\delta + n), \end{aligned} \tag{21}$$

$$F(x) = \alpha\Big[x - \frac{\pi}{s} g(x)\Big], \tag{22}$$

$$\phi(y) = \alpha \frac{s f^{-1}(\delta + n) - \pi(\delta + n)}{s}[\exp(y) - 1]. \tag{23}$$

In what follows, we shall denote the system of equations (19) and (20) with (21)-(23) by "System (PK*)." It is easy to see from Zeng et al. (1994) or Xiao and Zhang (2003) that System (PK*) can be viewed as a generalized Liénard system (under some assumptions). We shall look into System (PK*) to examine the characteristics of System (PK).

3.4 Existence of a periodic orbit

We shall now explore the existence of a periodic orbit, which can be taken as a persistent growth cycle, in System (PK*). For this purpose, we shall first confirm the local asymptotic total instability of the unique equilibrium,[9] second find a (nonempty) positively invariant compact subset of $\mathbb{R}^2$ (with respect to System (PK*))[10] and finally

[8] For details on generalized Liénard systems, see, for example, Levinson and Smith (1942), Zeng et al. (1994) or Xiao and Zhang (2003).

[9] We mean by the term "local asymptotic total instability" that the equilibrium point under consideration is either an unstable node or an unstable focus, i.e., that the trace and determinant of the Jacobian matrix evaluated at this equilibrium point are both positive.

[10] A closed (usually compact) region D is said to be positively invariant with respect to the dynamical system under consideration if every positive semi-trajectory (i.e., every solution path for $t \geq 0$) of this system which starts at an arbitrary point in D will remain in

apply the Poincaré-Bendixson theorem to System (PK*).[11]

To begin, we shall examine the local asymptotic stability of the unique equilibrium point of System (PK*), $(x^*y^*) = (0,0)$. One can easily derive the Jacobian matrix of System (PK*) evaluated at this equilibrium, denoted by J^*, as follows:

$$J^* = \begin{pmatrix} -F'(0) & \phi'(0) \\ -g'(0) & 0 \end{pmatrix} = \begin{pmatrix} \alpha[\pi f'(f^{-1}(\delta+n)) - s]/s & \alpha[sf^{-1}(\delta+n) - \pi(\delta+n)]/s \\ -f'(f^{-1}(\delta+n)) & 0 \end{pmatrix}.$$

The trace and determinant of J^* are given by

$$\begin{aligned} \operatorname{tr} J^* &= \alpha \frac{\pi f'(f^{-1}(\delta+n)) - s}{s}, \\ \det J^* &= \alpha f'(f^{-1}(\delta+n)) \frac{sf^{-1}(\delta+n) - \pi(\delta+n)}{s} > 0, \end{aligned}$$

where the inequality follows from (14) (derived from Assumption 2). Then, the unique equilibrium point of System (PK*) is not a saddle point. Also, it is locally asymptotically totally unstable if the trace of J^* is positive.

For the local asymptotic total instability of the unique equilibrium, we shall make the following assumption.

Assumption 4. *The following condition is satisfied:*

$$f'(f^{-1}(\delta+n)) > \frac{s}{\pi}. \tag{24}$$

Condition (24) holds if the rate of capital formation f is sufficiently elastic to a change in the expected rate of profit r^e at the unique equilibrium. In this respect, Assumption 4 may be said to reflect the argument by Keynes (1936, chap. 22) that investment is subjected to violent fluctuations due to changes in the marginal efficiency of capital, which is represented as the expected rate of profit r^e, even in short run. Moreover, we can make some remark on this assumption in terms of the so-called Keynesian stability condition (cf. Marglin and Bhaduri 1990).[12] In our System (PK) or (PK*), the Keynesian stability condition is always satisfied because the marginal propensity to invest with respect to the *current* income is zero. As we have observed, however, this does not imply the local asymptotic stability of the unique equilibrium. This consequence means that if investment is influenced not by the current rate of profit or of utilization but by the expected one, which is formed on the basis of the past and current ones, the Keynesian stability condition is not directly related to the stability of equilibrium.[13] Note that Assumption 4 implies that $F'(0) < 0$.

D for ever after (i.e., for all $t \geq 0$).

[11] For the Poincaré-Bendixson theorem, see, for example, Coddington and Levinson (1955, chap. 16).

[12] The Keynesian stability condition requires that the marginal propensity to invest be less than that to save.

[13] This consequence may be consistent with Skott's (2012) argument because he pointed out from Harrod's (1939) viewpoint that, if the rate of capital formation is formalized as a function not only of the current rate of utilization but also of the past ones (as well as the past rates of capital formation), the Keynesian stability condition does not have implications for the long run stability. For recent debates on the Keynesian stability condition, see, for example, Hein et al. (2011) or Franke (2017).

Next, we shall look for a (nonempty) positively invariant compact subset of $\mathbb{R}^2$. For this purpose, we shall draw the phase diagram of System (PK*). It follows from (19) and (20) that the loci of $\dot{x} = 0$ and of $\dot{y} = 0$ are given by $\phi(y) = F(x)$ and by $g(x) = 0$, which can be reduced to $x = 0$, respectively. To illustrate the phase diagram, we shall introduce the following assumption about f.

Assumption 5. *The following condition is satisfied:*

$$\lim_{r^e \to \infty} [sr^e - \pi f(r^e)] > sf^{-1}(\delta + n) - \pi(\delta + n). \tag{25}$$

Furthermore, there exist exactly two real constants $\underline{r}^e$ and $\overline{r}^e$ with $\underline{r}^e < \overline{r}^e$ such that the following conditions are satisfied:

$$f'(\underline{r}^e) = f'(\overline{r}^e) = \frac{s}{\pi}, \tag{26}$$

$$f(\overline{r}^e) < \frac{s}{\pi}\overline{r}^e. \tag{27}$$

Mathematically, condition (25) is fulfilled if the capital formation function f is bounded, and condition (26) holds if f is a logistic (sigmoid) function (under Assumption 4).[14] In this sense, these conditions are not so restrictive. Moreover, as we shall see below, condition (27) ensures that the locus of $\dot{x} = 0$ is well-defined.

We shall examine characteristics of the locus of $\dot{x} = 0$ by making use of Assumption 5 (as well as Assumptions 1-4). To this end, we shall first take a look at the shape of the graph of F. Noting (11), (17) and (22), we can obtain the following:

$$\lim_{x \to \infty} F(x) > 0, \tag{28}$$

Also, it follows from Assumptions 3 and 4 that

$$F(-r^*) = -\frac{\alpha}{s}\left[sf^{-1}(\delta + n) - (\delta + n) + sf(0)\right] < 0, \tag{29}$$

$$F'(0) = \alpha\frac{\pi}{s}\left[\frac{s}{\pi} - f'(f^{-1}(\delta + n))\right] < 0. \tag{30}$$

Since we know $F(0) = 0$ (from (22)), we can find from (30) that, for some sufficiently small positive ε, we have $F(-x) > 0$ and $F(x) < 0$ for $x \in (0, \varepsilon)$. Taking (28) and (29) into consideration, we can see from the continuity of $F'(x)$ (ensured by Assumption 1) that $F'(x) = 0$ has at least two roots $\underline{x}' \in (-r^*, 0)$ and $\overline{x}' > 0$. As for $F'(x)$, on the other hand, condition (26) in Assumption 5 is equivalent to meaning that $F'(x) = 0$ has exactly two real roots. Hence, $\underline{x}'$ and $\overline{x}'$ are all the real roots of $F'(x) = 0$ and given by $\underline{x}' = \underline{r}^e - r^* \in (-r^*, 0)$ and $\overline{x}' = \overline{r}^e - r^* > 0$. It then follows from the continuity of $F'(x)$ and (30) that $F'(x) < 0$ for $x \in (\underline{x}', \overline{x}')$ and $F'(x) > 0$ for $x \in (-r^*, \underline{x}')$

[14] For a related model using a logistic capital formation function, see Murakami (2018).

or $x > \overline{x}'$ and that $F(\underline{x}') > 0$ and $F(\overline{x}') < 0$. Furthermore, it is obvious from the continuity of $F(x)$ that $F(x) = 0$ has two roots $\underline{x}_0$ and $\overline{x}_0$ besides $x = 0$ with $\underline{x}_0 \in (-r^*, \underline{x}')$ and $\overline{x}_0 > \overline{x}'$. Thus, we can draw the graph of $F(x)$ as in figure 1.

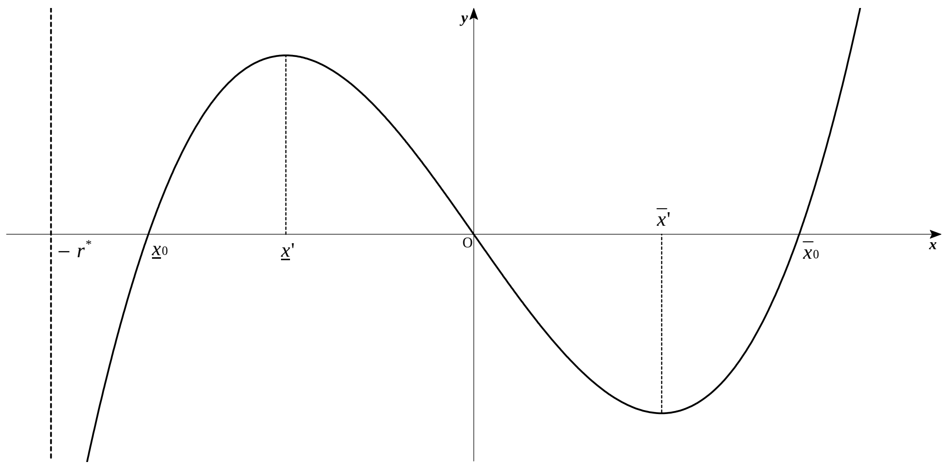

Figure 1: Graph of $F(x)$

We shall now have a closer look at the locus of $\dot{x} = 0$. It is easy to see from (22) and (23) that this locus is given by

$$\alpha \frac{sf^{-1}(\delta+n) - \pi(\delta+n)}{s}[\exp(y) - 1] = \alpha\Big[x - \frac{\pi}{s}g(x)\Big],$$

or

$$\exp(y) = 1 + \frac{sx - \pi g(x)}{sf^{-1}(\delta+n) - \pi(\delta+n)}. \tag{31}$$

Then, for the locus of $\dot{x} = 0$ to be well defined at least for $x \geq \underline{x}_0$, it is necessary and sufficient that the right hand side of (31) is positive for $x \geq \underline{x}_0$.[15] Since we know from figure 1 that the minimum of $F(x)$ for $x \geq \underline{x}_0$ is given by $F(\overline{x}')$, it suffices for the locus of $\dot{x} = 0$ being well defined that the following condition is fulfilled:

$$1 + \frac{s\overline{x}' - \pi g(\overline{x}')}{sf^{-1}(\delta+n) - \pi(\delta+n)} > 0,$$

which can, by (21), be reduced to

$$\frac{s\overline{r}^e - \pi f(\overline{r}^e)}{sf^{-1}(\delta+n) - \pi(\delta+n)} > 0.$$

This condition is satisfied due to (14) (by Assumption 1) and (27). Thus, the locus of $\dot{x} = 0$ is well defined at least

[15]For our analysis, it is sufficient that the locus of $\dot{x} = 0$ is well defined for $x \geq \underline{x}_0$. This requirement is vital for verifying the existence and uniqueness of a limit cycle in System (PK*).

for $x \geq \underline{x}_0$.

Therefore, the phase diagram of System (PK*) can be drawn as in figure 2.

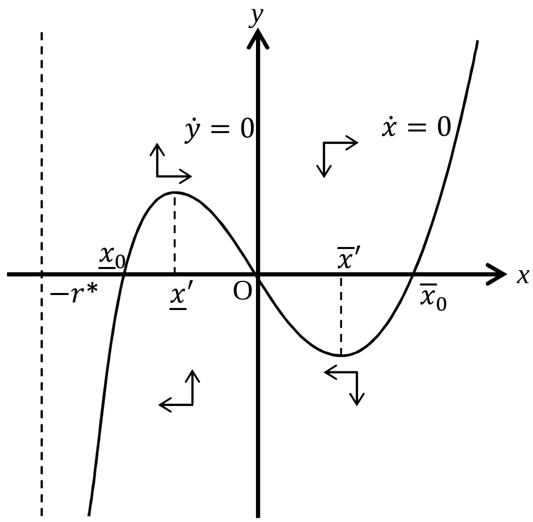

Figure 2: Phase diagram of System (PK*)

Now we are in a position to detect a (nonempty) positively invariant compact subset of $\mathbb{R}^2$. To begin, we shall consider the solution path of System (PK*) with $(x(0), y(0)) = (-r^*, 0)$, denoted by (SP1).[16] We shall below prove that (SP1) reaches the y-axis on the positive part at some (finite) nonnegative t (for the first time). To this end, we shall first confirm that $\dot{x} > 0$ along (SP1) before it intersects with the y-axis on the positive part for the first time.[17] It is seen from (19) and (20) that for $x < 0$, along the locus of $\dot{x} = 0$,

$$\frac{d}{dt}\dot{x}|_{\dot{x}=0} = [\phi'(y)\dot{y} - F'(x)\dot{x}]|_{\dot{x}=0} = -g(x)\phi'(y) > 0,$$

where the inequality follows from the fact that $g(x) < 0$ for $x < 0$. It then follows from the continuity of solution paths that solution paths of System (PK*) cannot cross the locus of $\dot{x} = 0$, if they stay on the domain of $x < 0$ and start on the domain of $\dot{x} > 0$. Thus, we can confirm the desirable fact because $(x(0), y(0)) = (-r^*, 0)$ lies on the domain of $\dot{x} > 0$ (cf. Assumption 3).

We shall next verify that (SP1) crosses the line of $x = \underline{x}'$ from left to right at some nonnegative t. For this purpose, we shall suppose, for the sake of contradiction, that (SP1) stays on the domain of $x \in [-r^*, \underline{x}']$ for all $t \geq 0$.[18] Then, we have $g(x) \leq g(\underline{x}') < 0$ and $F(x) \leq F(\underline{x}')$ for $x \in [-r^*, \underline{x}']$ because $g'(x) > 0$ and $F'(x) > 0$ for $x \in [-r^*, \underline{x}']$. Hence, we can obtain the following inequalities for all $t \geq 0$ along (SP1) under our hypothetical

[16]Note that the existence or uniqueness of (SP1) is not guaranteed at this stage. In what follows, we shall suppose that (SP1) exists until it reaches the y-axis on the positive part for the first time, discuss characteristics of an arbitrary (SP1) and then establish the existence and uniqueness of it.

[17]Note that, at this stage, we do not rule out the possibility that (SP1) never reaches the y-axis for $t \geq 0$.

[18]Note that, by Assumption 3, solution paths of System (PK) cannot leave the domain of $x \geq -r^*$ once they enter this domain.

assumption:

$$\dot{x} \geq \phi(y) - F(\underline{x}'), \tag{32}$$

$$\dot{y} \geq -g(\underline{x}') > 0. \tag{33}$$

Also, because of $\phi(0) = 0,\ \phi'(y) > 0$ and $\phi(y) \to \infty$ as $y \to \infty$ due to (23) and of $F(\underline{x}') > 0$, there exists a unique positive y_1 such that $\phi(y_1) = F(\underline{x}') + 1$. It follows from (33) that along (SP1)

$$y(t) \geq y(0) - g(\underline{x}')t = -g(\underline{x}')t.$$

Hence, we have $y(t) \geq y_1$ for $t \geq t_1 \equiv -y_1/g(\underline{x}') > 0$ along (SP1). Then, we can find from (32) that for $t \geq t_1$

$$\dot{x} \geq \phi(y) - F(\underline{x}') \geq \phi(y_1) - F(\underline{x}') = 1.$$

Since (SP1) stays on the domain of $x \in [-r^*, \underline{x}']$ for all $t \geq 0$ (by assumption), we have $x(t_1) \geq -r^*$ and

$$\dot{x}(t) \geq x(t_1) + (t - t_1) \geq -r^* + (t - t_1).$$

Then, we have $x(t) > \underline{x}'$ for $t > t_2 \equiv t_1 + (\underline{x}' + r^*)$. But this contradicts our hypothesis. Thus, we can prove that (SP1) must cross the line of $x = \underline{x}'$ from left to right by the time of $t = t_2$.

We shall then show that (SP1) reaches the y-axis on the positive part after it crosses the line of $x = \underline{x}'$ at $t = t_3$ (where $t_3 \leq t_2$). Since $\dot{x} > 0$ along (SP1) before it reaches the y-axis on the positive part, we have $x(t) \geq \underline{x}'$ for $t \geq t_3$ before it reaches the y-axis. Thus, we shall assume, for the sake of contradiction, that (SP1) stays on the domain of $x \in [\underline{x}', 0)$ for all $t \geq t_3$. Then, (SP1) remains on the domain of $x \in [\underline{x}', 0]$ (but never reaches the y-axis) for $t \geq t_3$. Since $\dot{x} > 0$ at $t = t_3$ along (SP1), we have $\dot{x}(t_3) \equiv \Delta > 0$. Because $F'(x) \leq 0$ and $\dot{y} = -g(x) > 0$ for $x \in [\underline{x}', 0]$ (cf. figure 1), we can see that along (SP1), for $t \geq t_3$

$$\frac{d}{dt}\dot{x} = \phi'(y)\dot{y} - F'(x)\dot{x} \geq 0.$$

Then, we can see that, along (SP1), for $t \geq t_3$

$$\dot{x}(t) \geq \dot{x}(t_3) = \Delta,$$

or

$$x(t) \geq x(t_3) + \Delta(t - t_3) = \underline{x}' + \Delta(t - t_3).$$

Therefore, (SP1) must reach the y-axis (on the positive part) by the time of $t = t_4 \equiv t_3 - \underline{x}'/\Delta$. This is, of course, a contradiction. Hence, (SP1) can be shown to reach the y-axis on the positive part by the time of $t = t_4$ after it crosses the line of $x = \underline{x}'$.

We shall now make sure that (SP1) uniquely exists at least until it reaches the y-axis for the first time. Because of $0 \leq \dot{y} \leq -g(r^*)(> 0)$ along (SP1) for $t \in [0, t_4]$, we have $0 \leq y(t) \leq y_\# = -g(r^*)t_4$ for $t \in [0, t_4]$, along (SP1). Then, every (SP1), if it exists, stays on the following (nonempty) compact rectangular domain D_1 for $t \in [0, t_4]$:

$$D_1 \equiv \{(x, y) \in \mathbb{R}^2 : \ x \in [-r^*, 0], \ y \in [0, y_\#]\}.$$

Since g, F and ϕ are all continuously differentiable on D_1 (by Assumption 16) and D_1 is a compact convex subset of $\mathbb{R}^2$, it follows from the mean value theorem that there exists a positive constant M such that for every (x', y'), $(x'', y'') \in D_1$[19]

$$|[\phi(y'') - F(x'')] - [\phi(y') - F(x')]| + |g(x'') - g(x')| \leq M(|x'' - x'| + |y'' - y'|).$$

Hence, System (PK*) satisfies the Lipschitz condition (cf. Coddington and Levinson 1955, chap. 1) on D_1 (for all $t \geq 0$). It then follows from the argument on continuation of solution paths (cf. Coddington and Levinson 1955, p. 15, Theorem 4.1) that the existence and uniqueness of a solution path of System (PK*) can be obtained as long as it stays on D_1. Therefore, (SP1) exists and is unique until it reaches the y-axis for the first time. Let $\mathrm{Q}(0, y_q)$ be the (first) intersection point of the unique (SP1) and the y-axis on the positive part, where $0 < y_q \leq y_\#$. Also, let $\mathrm{R}(x_r, y_q)$ be the intersection point of the locus of $\dot{x} = 0$ and the line through Q parallel with the x-axis.[20]

We shall next consider the solution path of System (PK*) with $(x(0), y(0)) = (x_r, 0)$, denoted by (SP2). By a method similar to the above, we can verify that (SP2) exists and is unique (at least) until it reaches the y-axis on the negative part for the first time and that it must reach the y-axis on the negative part at some finite nonnegative time t. Let $\mathrm{T}(0, y_t)$ be the (first) intersection point of (SP2) and the y-axis on the negative part, where y_t is a negative constant.

Now we can proceed to detect a (nonempty) positively invariant compact subset of $\mathbb{R}^2$. Define the points $\mathrm{P}(-r^*, 0)$, $\mathrm{S}(x_r, 0)$ and $\mathrm{U}(-r^*, y_t)$ and let the arcs of PQ and ST represent (SP1) and (SP2), discussed above, respectively. Then, we can construct the (nonempty) compact subset of $\mathbb{R}^2$ enclosed by the arcs of PQ and ST and the line segments of QR, RS, TU and UP, as in figure 3. We shall denote this subset by D. We can see that D is positively invariant as follows. By the uniqueness of (SP1) and (SP2), no solution paths of System (PK*) which start from inside of D at $t = 0$ can cross the arc of PQ or of ST. Also, we have $\dot{y} \leq 0$ along QR, $\dot{x} \leq 0$ along RS, $\dot{y} \geq 0$ along TU and $\dot{x} > 0$ along UP. Then, no solution paths of System (PK*) can leave D if they enter this

[19]Murakami (2014) proved in details that the Lipschitz condition is satisfied on a compact rectangular subset of $\mathbb{R}^n$ if all the functions are continuously differentiable on this subset.

[20]Since the locus of $\dot{x} = 0$ is given by (31), we can find from (28) (and figure 1) that such a positive x_r exists and is unique.

domain.

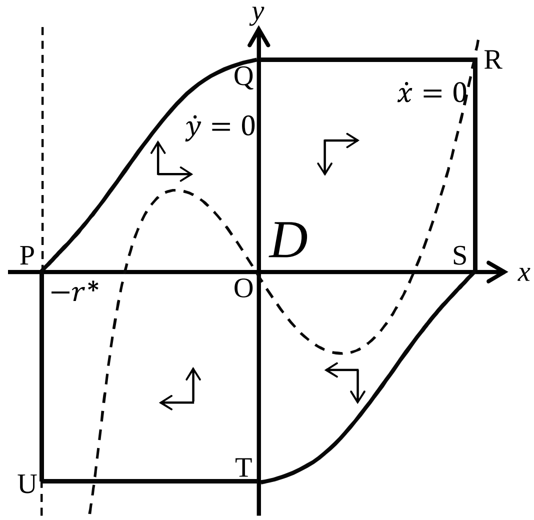

Figure 3: Positively invariant subset D

Thus, we are ready to establish the existence of a periodic orbit in System (PK) or (PK*) by way of the Poincaré-Bendixson theorem.

Proposition 1. *Let Assumptions 1-5 hold. Then, System (PK) has at least one periodic orbit on* $\mathbb{R}^2_{++}$.

Proof. To prove this proposition, it suffices to verify the existence of a periodic orbit which lies entirely on the domain of $x > -r^*$ in System (PK*).

To apply the Poincaré-Bendixson theorem (cf. Coddington and Levinson 1955, chap.16), we shall first confirm the uniqueness of a solution path of System (PK*) on D.[21] For this purpose, we shall construct the (nonempty) compact rectangular domain D_2 as follows:

$$D_2 \equiv \{(x, y) \in \mathbb{R}^2 : \; x \in [-r^*, x_r], \; y \in [y_t, y_q]\},$$

where $\mathrm{Q}(0, y_q)$, $\mathrm{R}(x_r, y_q)$ are $\mathrm{T}(0, y_t)$ are defined above.[22] Since D_2 is a a compact rectangular domain, we can verify in the same way as above that System (PK*) satisfies the Lipschitz condition on D_2. Then, every solution path of System (PK*) is unique (with respect to an initial condition) provided that it stays on D_2. Hence, the uniqueness of a solution path follows on D.

Since the unique equilibrium point of System (PK*) is locally asymptotically totally unstable (by Assumption 4), we can enclose the equilibrium point by a sufficiently small rectangle such that no solution paths of System (PK*) can enter the interior of this rectangle once they leave it. We can also construct a (nonempty) positively

[21]Since g, F and ϕ are all continuously differentiable and D is a positively invariant compact subset of $\mathbb{R}^2$, we can prove that if $(x(0), y(0)) \in D$, the solution path of (PK*) exists on D for all $t \geq 0$. (cf. Coddington and Levinson 1955, chap. 1).
[22]As we have seen, these points are uniquely determined.

invariant compact domain D^* by eliminating the interior of the small rectangle from D. Because the positively invariant domain D^* contains no equilibrium point, we can apply the Poincaré-Bendixson theorem (cf. Coddington and Levinson 1955, chap.16) to System (PK*) on D^* to draw the conclusion that a periodic orbit of System (PK*) exists on D^*. Due to $\dot{x} > 0$ along the line of $x = -r^*$ (by Assumption 3), such a periodic orbit cannot intersect with this line and lies entirely on the domain of $x > -r^*$. □

Proposition 1 guarantees the existence of a periodic orbit in our post Keynesian System (PK). This implies that a persistent and periodic "growth cycle" can be observed in System (PK). Along such a growth cycle, the expected rate of profit r^e and per capita capital stock k are subjected to persistent cyclical fluctuations, and so is the output-capital ratio u due to (5). Since the size of population N changes at a constant rate, aggregate capital stock $K = kN$ fluctuates around its trend value $k^*N(t)$, and so aggregate income $Y = ukN(t)$ also fluctuates around its trend value $u^*k^*N(t)$, where u^* is obtained by putting $r^e = r^*$ and $k = k^*$ in (5). In this respect, our growth cycle may be said to describe economies growing (or shrinking) over persistent cyclical fluctuations.

We shall add some remarks on growth cycles in our System (PK) from a Keynesian viewpoint. As is well known, Keynes (1936, chap. 22) attributed the main cause of business cycles to violent fluctuations of the marginal efficiency of capital, which can be identified with the expected rate of profit r^e. Our conclusion on the presence of persistent growth cycles is consistent with his argument because changes in r^e do give rise to persistent cyclical economic fluctuations. In this sense, it may be maintained that our post Keynesian system (PK) can properly describe the phenomenon of business cycles based upon the Keynesian theory.[23]

3.5 Uniqueness of a limit cycle

Now that the existence of a periodic orbit is ensured in System (PK), we shall examine the "uniqueness" of it. If this uniqueness obtains, the unique periodic orbit does represent the only (feasible) growth cycle in our system (and mathematically, it is a limit cycle).[24] To derive a sufficient condition for the uniqueness, we shall continue to investigate properties of System (PK*) instead of those of System (PK). Specifically, we shall make use of the theorem established by Xiao and Zhang (2003), which is reproduced as Theorem 1 in Appendix.[25]

We can easily find that Assumptions 6 and 7, required in Theorem 1, are fulfilled in System (PK*), by setting $\underline{x} = -r^* = -f^{-1}(\delta+n)$, $\overline{x} = \infty$, $\underline{y} = -\infty$ and $\overline{y} = \infty$ (if $\underline{x}_0$ and $\overline{x}_0$ are defined as in the previous section). Thus, to

[23]Our consequence is consistent with Benassy's (1984) one in that a persistent business cycle (or a periodic orbit) can be generated if the effect of changes in the expected level of aggregate demand is strong enough. Note that his analysis was a short-term one, while ours is a long-term one.

[24]In economics, a limit cycle is often confused with a periodic orbit. Mathematically, however, not every periodic orbit is a limit cycle. Indeed, a limit cycle is a periodic orbit such that some other solution path has it as an α or ω limit set, i.e., that the former converges to the latter as $t \to \infty$ or as $t \to -\infty$. (cf. Coddington and Levinson 1955, pp. 391-392; Hirsch and Smale 1974, p. 250). For example, all solution paths, except for those whose initial condition is an equilibrium point, of Lotka-Volterra systems, which Goodwin (1967) employed in presenting a model of growth cycle, are periodic orbits but none of them is a limit cycle (cf. Hirsch and Smale 1974, p. 252, Theorem 3). If the number of periodic orbits is finite, on the other hand, all of them are necessarily (either stable or unstable) limit cycles (on two-dimensional differential equations).

[25]Precisely speaking, Theorem 1 is a slightly modified version of Xiao and Zhang's (2003) theorem, which was presented by Murakami (2018).

guarantee the uniqueness of a limit cycle, we only have to examine under what condition the remaining hypothesis for Theorem 1, or Assumption 8, is satisfied.

Now we shall see if Assumption 8 is fulfilled in System (PK*). As we have confirmed in the previous section, the locus of $\phi(y) = F(x)$, given by (31), is well defined at least on $[\underline{x}_0, \overline{x}_0]$ (by Assumption 5). Then, in what follows, we shall detect a sufficient condition for condition (44) to hold. Since we have $\underline{x}' \in [\underline{x}_0, 0]$ and $\overline{x}' \in [0, \overline{x}_0]$, condition (44) holds if the following condition is fulfilled:[26]

$$\begin{cases} G(\overline{x}') + \Phi(\phi^{-1}(F(\overline{x}'))) \geq G(\underline{x}_0), \text{ if } G(\underline{x}_0) \geq G(\overline{x}_0), \\ G(\underline{x}') + \Phi(\phi^{-1}(F(\underline{x}'))) \geq G(\overline{x}_0), \text{ if } G(\overline{x}_0) > G(\underline{x}_0), \end{cases}$$

or

$$\begin{cases} \Phi(\phi^{-1}(F(\overline{x}'))) \geq G(\underline{x}_0) - G(\overline{x}'), \text{ if } G(\underline{x}_0) \geq G(\overline{x}_0), \\ \Phi(\phi^{-1}(F(\underline{x}'))) \geq G(\overline{x}_0) - G(\underline{x}'), \text{ if } G(\overline{x}_0) > G(\underline{x}_0). \end{cases} \tag{34}$$

One can easily see that the inverse of ϕ is given by (23) as

$$\phi^{-1}(y) = \ln\Big(1 + \frac{sy}{sf^{-1}(\delta+n) - \pi(\delta+n)}\Big).$$

It then follows from (22) that

$$\phi^{-1}(F(x)) = \ln\Big(1 + \frac{sx - \pi g(x)}{sf^{-1}(\delta+n) - \pi(\delta+n)}\Big). \tag{35}$$

On the other hand, we can easily calculate the following:

$$\Phi(y) = \int_0^y \phi(s)ds = \alpha \frac{sf^{-1}(\delta+n) - \pi(\delta+n)}{s}[\exp(y) - y - 1]. \tag{36}$$

By substituting (35) in (36), we obtain

$$\begin{aligned} \Phi(\phi^{-1}(F(x))) &= \alpha \frac{sf^{-1}(\delta+n) - \pi(\delta+n)}{s}\Big[\frac{sx - \pi g(x)}{sf^{-1}(\delta+n) - \pi(\delta+n)} - \ln\Big(1 + \frac{sx - \pi g(x)}{sf^{-1}(\delta+n) - \pi(\delta+n)}\Big)\Big] \\ &= \alpha \frac{sf^{-1}(\delta+n) - \pi(\delta+n)}{s}[z - \ln(1+z)], \end{aligned} \tag{37}$$

where $z = [sx - \pi g(x)]/[sf^{-1}(\delta+n) - \pi(\delta+n)]$. Since $sx - \pi g(x) = sF(x)/\alpha$ is not equal to 0 for $x = \underline{x}'$ or for $x = \overline{x}'$ (cf. figure 1) and we have $z > \ln(1+z)$ for $z \neq 0$ (provided that $z > -1$), we can immediately find from (37) that $\Phi(\phi^{-1}(F(\underline{x}')))$ and $\Phi(\phi^{-1}(F(\overline{x}')))$ are both positive. Moreover, $\underline{x}'$ and $\overline{x}'$ are both determined independently from the value of α because they are roots of $F'(x) = 0$ or $g'(x) = s/\pi$. Hence, the left hand side of (34) is positive

[26]Because of $\phi'(y) > 0$ (cf. (23)), the function $\phi^{-1}(F(x))$ can be defined at least for $x \in [\underline{x}_0, \overline{x}_0]$ (by Assumption 5).

and proportionate to α in both cases. On the other hand, the right hand side of (34) is independent from α because $\underline{x}_0$ and $\overline{x}_0$ are both determined, independently from α, by $F(x) = 0$ or $sx = \pi g(x)$. Therefore, one can, by taking sufficiently large α, make the left hand side of (34) as large as one likes while fixing the value of the right hand side. Thus, we can argue that condition (34) is satisfied for α sufficiently large.

Now we are ready to present the following pleasant conclusion on the uniqueness of a limit cycle.

Proposition 2. *Let Assumptions 1-5 hold. If α is sufficiently large, System (PK) has a unique limit cycle on $\mathbb{R}^2_{++}$, and it is (periodically) stable.*

Proof. Since we have already shown in Proposition 1 the existence of a periodic orbit on $\mathbb{R}^2_{++}$ in System (PK), it suffices to prove the uniqueness and stability of a limit cycle (or of a periodic orbit) in System (PK*). As we have observed above, Assumptions 6-8 are fulfilled in System (PK*) for α sufficiently large. Then, we can obtain the conclusion of this proposition. □

According to Proposition 2, the uniqueness (and stability) of a limit cycle obtains in System (PK) if α is large enough. Put in a different way, the state of the economy under consideration, represented by (r^e, k), converges to the unique growth cycle with the passage of time (provided that the initial condition of (r^e, k) is not (r^*, k^*) and that variations in these variables are bounded over time).[27] This is a strong conclusion in that if the revision speed of expectations α is quick enough, the asymptotic state of the (Keynesian) macroeconomic system is the *uniquely* determined growth cycle (unless the initial condition is not the equilibrium state) and so it is predictable.

We shall add some comments on the conclusion of Proposition 2. The uniqueness of a limit cycle is itself a "unique" consequence in that it has rarely been verified in economics.[28] Furthermore, this consequence is significant from a Keynesian perspective. As we have confirmed, Keynes (1936) himself emphasized sharp variations in the marginal efficiency of capital, which is conceptually identical with the expected rate of profit r^e, as the primary factor for occurrence of business cycles. In our analysis, on the other hand, if the speed of revision of expectation α is large, which means that r^e is exposed to violent fluctuations (cf. (8)), the uniqueness of a growth cycle obtains. Therefore, we may state from our present analysis that frequent and violent variations in the marginal efficiency of capital compose a sufficient condition not only for the *existence* of a business cycle, as Keynes (1936) insisted, but also for the *uniqueness* of it.

4 Conclusion

We shall now summarize our analysis and conclude the present paper.

[27] Proposition 2 only ensures the existence and uniqueness of a limit cycle and not the convergence of every solution path to the unique limit cycle. However, it is easy to obtain this convergence by making use of the generalized Poincaré-Bendixson theorem (cf. Coddington and Levinson 1955, pp. 394-395, Theorem 3.1) unless the solution path is unbounded.

[28] There have been several exceptions such as Ichimura (1955), Lorenz (1988, 1993), Gori and Galeotti (1989), Sasakura (1996) and Murakami (2018).

We have formalized a post Keynesian system by introducing the investment function as related directly to the expected rate of profit and examined the existence and uniqueness of a periodic orbit (or of a limit cycle) in this system. Compared with the other related Keynesian systems, our post Keynesian system may be said to conform more to Keynes' idea that expectation, especially the long-term expectation, has a decisive influence on the determination of investment and aggregate income. Importantly, we have demonstrated that a periodic orbit does exist if a change in the expected rate of profit has a strong influence on investment and that it is unique if revisions of the expected rate of profit are frequent enough (more specifically if the speed of revision of expectation is sufficiently fast). By so doing, we have been able to confirm that variations in expectation do give rise to a persistent business cycle and contribute to the uniqueness of it. Our conclusion may be said to support and strengthen Keynes' thought on business cycles.

Throughout this paper, we have attempted to shed a new light on the role of expectation in business cycles. Although the importance of expectation has been recognized, it has not properly been examined in theory of business cycles. We hope sincerely that our present analysis is helpful for understanding the macroeconomic system.

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Appendix: Generalized Liénard systems

We shall introduce the theorem by Xiao and Zhang (2003) on the uniqueness of a (stable) limit cycle in generalized Liénard systems.

We shall consider the following generalized Liénard system:

$$\dot{x} = \phi(y) - F(x), \tag{38}$$

$$\dot{y} = -g(x). \tag{39}$$

In what follows, the system of equations (38) and (39) is denoted by "System (L)."

Following Xiao and Zhang (2003), we shall impose the following assumptions concerning System (L).

Assumption 6. *The real valued functions $g(x)$ and $F(x)$ are, respectively, continuous and continuously differentiable on $(\underline{x}, \overline{x})$, and the real valued function $\phi(y)$ is continuously differentiable on $(\underline{y}, \overline{y})$ with $-\infty \leq \underline{x} < 0 < \overline{x} \leq \infty$*

and $-\infty \leq \underline{y} < 0 < \overline{y} \leq \infty$. *Furthermore, the following conditions are satisfied:*

$$xg(x) > 0 \text{ for } x \neq 0, \tag{40}$$

$$\phi(0) = 0,\ \ \phi'(y) > 0 \text{ for } y \in (\underline{y}, \overline{y}). \tag{41}$$

Assumption 7. *There exist* $\underline{x}_0$ *and* $\overline{x}_0$ *with* $\underline{x} < \underline{x}_0 < 0 < \overline{x}_0 < \overline{x}$ *such that the following conditions are satisfied:*

$$F(\underline{x}_0) = F(0) = F(\overline{x}_0) = 0, \tag{42}$$

$$\begin{cases} xF(x) \leq 0 \text{ for } x \in (\underline{x}_0, \overline{x}_0), \\ xF(x) > 0,\ \ F'(x) \geq 0 \text{ for } x \in (\underline{x}, \underline{x}_0) \text{ or } x \in (\overline{x}_0, \overline{x}). \end{cases} \tag{43}$$

Furthermore, $F(x)$ *is not identically equal to 0 for* x *sufficiently close to 0.*

Assumption 8. *The curve of* $\phi(y) = F(x)$ *is well defined for* $x \in [\underline{x}_0, \overline{x}_0]$.[29] *Furthermore, the following condition is satisfied:*

$$\begin{cases} \sup_{x\in[0,\overline{x}_0]}(G(x) + \Phi(\phi^{-1}(F(x)))) \geq G(\underline{x}_0),\ \text{ if } G(\underline{x}_0) \geq G(\overline{x}_0), \\ \sup_{x\in[\underline{x}_0,0]}(G(x) + \Phi(\phi^{-1}(F(x)))) \geq G(\overline{x}_0),\ \text{ if } G(\overline{x}_0) > G(\underline{x}_0), \end{cases} \tag{44}$$

where $G(x) = \int_0^x g(s)ds$ *and* $\Phi(y) = \int_0^y \phi(s)ds$.

As regards the uniqueness of a limit cycle in System (L), the following theorem was verified by Xiao and Zhang (2003).

Theorem 1. *Let Assumptions 6-8 hold. Then, System (L) has at most one limit cycle, and it is (periodically) stable if it exists.*

Proof. See Xiao and Zhang (2003, p. 1187, Theorem 2.2). □

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