高性能近似アルゴリズムの系統的設計法とその応用に関する研究

研究代表者　研　究　員　**浅野　孝夫**（理工学部情報工学科）
共同研究者　研　究　員　**築山　修治**（理工学部電気電子情報通信工学科）

1　はじめに

情報ネットワークや VLSI の物理設計等で生じる自然な問題は NP-困難であることが多く，厳密解を求めるのは長い計算時間を要する。そこで近似解を求めて利用することになるが，その際重要になるのが解の品質である。厳密解に匹敵する高品質な解を求める研究が近似アルゴリズムの研究であるが，最近数理計画法に基づく系統的設計法の有用性が注目を浴びてきている。一般に NP-困難な問題の多くは整数計画問題をして定式化できる。その整数条件を外して線形計画問題や半正定値計画問題に緩和して解き，その最適解の値を元の整数計画問題の最適解の値の下界あるいは上界として用いて，解の品質を保証するというものが，数理計画法に基づく近似アルゴリズム設計法である。緩和問題にすることにより，数理計画法の双対理論に基づいた手法が，近似アルゴリズムでも適用可能になり，従来の近似性能が最近大幅に改善されてきている。

本研究では，上記の研究背景に基づいて，高性能近似アルゴリズムの系統的設計法の有用性を明らかにすることを目的としている。この目的を達成するための研究を実行してきたが，今回は，情報科学の最も基本的な問題である最大充足化問題 (MAX SAT) に対して高性能アルゴリズムを与える。

2　MAX SAT の研究の概要

MAX SAT, one of the most well-studied NP-hard problems, is stated as follows: given a set of clauses with weights, find a truth assignment that maximizes the sum of the weights of the satisfied clauses. More precisely, an instance of MAX SAT is defined by $(\mathcal{C}, w)$, where $\mathcal{C}$ is a set of boolean clauses, each clause $C \in \mathcal{C}$ being a disjunction of literals and having a positive weight $w(C)$. Let $X = \{x_1, \ldots, x_n\}$ be the set of boolean variables in the clauses of $\mathcal{C}$. A *literal* is a variable $x \in X$ or its negation $\bar{x}$. For simplicity we assume $x_{n+i} = \bar{x}_i$ $(x_i = \bar{x}_{n+i})$. Thus, $\bar{X} = \{\bar{x} \mid x \in X\} = \{x_{n+1}, x_{n+2}, \ldots, x_{2n}\}$ and $X \cup \bar{X} = \{x_1, \ldots, x_{2n}\}$. We assume that no literals with the same variable appear more than once in a clause in $\mathcal{C}$. For each $x_i \in X$, let $x_i = 1$ $(x_i = 0$, resp.) if x_i is true (false, resp.). Then, $x_{n+i} = \bar{x}_i = 1 - x_i$ and a clause $C_j = x_{j_1} \vee x_{j_2} \vee \cdots \vee x_{j_{k_j}} \in \mathcal{C}$ can be considered to be a function

$$C_j = C_j(\boldsymbol{x}) = 1 - \prod_{i=1}^{k_j}(1 - x_{j_i})$$

on $\boldsymbol{x} = (x_1, \ldots, x_{2n}) \in \{0,1\}^{2n}$. Thus, $C_j = C_j(\boldsymbol{x}) = 0$ or 1 for any *truth assignment* $\boldsymbol{x} \in \{0,1\}^{2n}$ with $x_i + x_{n+i} = 1$ $(i = 1, 2, ..., n)$ and C_j is *satisfied* if $C_j(\boldsymbol{x}) = 1$. The *value* of a truth assignment $\boldsymbol{x}$ is defined to be

$$F_{\mathcal{C}}(\boldsymbol{x}) = \sum_{C_j \in \mathcal{C}} w(C_j) C_j(\boldsymbol{x}).$$

That is, the value of $\boldsymbol{x}$ is the sum of the weights of the clauses in $\mathcal{C}$ satisfied by $\boldsymbol{x}$. Thus, the goal of MAX SAT is to find an optimal truth assignment (i.e., a truth assignment of maximum value). We will also use MAX kSAT, a restricted version of the problem in which each clause has at most k literals.

MAX SAT is known to be NP-hard and many approximation algorithms for it have been proposed. Håstad [5] has shown that no approximation algorithm for MAX SAT can achieve performance guarantee better than 7/8 unless $P = NP$. On the other hand, Asano and Williamson [1] have presented a 0.7846-approximation algorithm and an approximation algorithm whose performance guarantee is 0.8331 if a conjectured performance guarantee of 0.7977 is true in the Zwick's algorithm [9]. Both algorithms are based on their sharpened analysis of Goemans and Williamson's LP-relaxation for MAX SAT [3].

In this paper, we present an improved analysis which is simpler than the previous analysis by Asano and Williamson [1]. Furthermore, we show that this analysis will lead to approximation algorithms with better performance guarantees if combined with other approximation algorithms which were (or will be) presented. Actually, algorithms based on this analysis lead to approximation algorithms with performance guarantee 0.7877 and conjectured performance guarantee 0.8353 which are slightly better than the best known corresponding performance guarantees 0.7846 and 0.8331 respectively, if combined with the MAX 2SAT and MAX 3SAT algorithms by Halperin and Zwick [6] and the Zwick's algorithm [9], respectively. Thus, algorithms based on this analysis will be used as a building block in designing an improved approximation algorithm for MAX SAT.

To explain our result in more detail, we briefly review the 0.75-approximation algorithm of Goemans and Williamson based on the probabilistic method [3]. Let $\boldsymbol{x}^p = (x_1^p, \ldots, x_{2n}^p)$ be a *random* truth assignment with $0 \le x_i^p = p_i \le 1$ ($x_{n+i}^p = 1 - x_i^p = 1 - p_i = p_{n+i}$). That is, $\boldsymbol{x}^p$ is obtained by setting independently each variable $x_i \in X$ to be true with probability p_i (and $x_{n+i} = \bar{x}_i$ to be true with probability $p_{n+i} = 1 - p_i$). Then the probability of a clause $C_j = x_{j_1} \vee x_{j_2} \vee \cdots \vee x_{j_{k_j}} \in \mathcal{C}$ satisfied by the random truth assignment $\boldsymbol{x}^p = (x_1^p, \ldots, x_{2n}^p)$ is

$$C_j(\boldsymbol{x}^p) = 1 - \prod_{i=1}^{k_j} (1 - x_{j_i}^p).$$

Thus, the expected value of the random truth assignment $\boldsymbol{x}^p$ is

$$F_{\mathcal{C}}(\boldsymbol{x}^p) = \sum_{C_j \in \mathcal{C}} w(C_j) C_j(\boldsymbol{x}^p).$$

The probabilistic method assures that there is a truth assignment $\boldsymbol{x}^q \in \{0,1\}^{2n}$ of value at least $F_{\mathcal{C}}(\boldsymbol{x}^p)$. Such a truth assignment $\boldsymbol{x}^q$ can be obtained by the method of conditional probabilities [3].

Using an IP (integer programming) formulation of MAX SAT and its LP (linear programming) relaxation, Goemans and Williamson [3] obtained an algorithm for finding a random truth assignment $\boldsymbol{x}^p$ of value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ at least

$$\sum_{k \ge 1} (1 - (1 - \frac{1}{k})^k) \hat{W}_k \ge (1 - \frac{1}{e}) \hat{W} \approx 0.632 \hat{W},$$

where e is the base of natural logarithm, $\hat{W}_k = \sum_{C \in \mathcal{C}_k} w(C) C(\hat{\boldsymbol{x}})$, and $F_{\mathcal{C}}(\hat{\boldsymbol{x}}) = \sum_{k \ge 1} \hat{W}_k$ for an optimal truth assignment $\hat{\boldsymbol{x}}$ of $(\mathcal{C}, w)$ ($\mathcal{C}_k$ denotes the set of clauses in $\mathcal{C}$ with k literals). Goemans and Williamson also obtained a 0.75-approximation algorithm by using a hybrid approach of combining the above algorithm with Johnson's algorithm [7]. It finds a random truth assignment of value at least

$$\begin{aligned} &0.750\hat{W}_1 + 0.750\hat{W}_2 + 0.789\hat{W}_3 + 0.810\hat{W}_4 \\ &+0.820\hat{W}_5 + 0.824\hat{W}_6 + \textstyle\sum_{k \ge 7} \beta_k \hat{W}_k, \end{aligned}$$

where

$$\beta_k = \frac{1}{2}\left(2 - \frac{1}{2^k} - \left(1 - \frac{1}{k}\right)^k\right).$$

Asano and Williamson [1] showed that one of the non-hybrid algorithms of Goemans and Williamson finds a random truth assignment $\boldsymbol{x}^p$ with value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ at least

$$\begin{aligned} &0.750\hat{W}_1 + 0.750\hat{W}_2 + 0.804\hat{W}_3 + 0.851\hat{W}_4 \\ &+0.888\hat{W}_5 + 0.915\hat{W}_6 + \textstyle\sum_{k \ge 7} \gamma_k \hat{W}_k, \end{aligned}$$

where

$$\gamma_k = 1 - \frac{1}{2}\left(\frac{3}{4}\right)^{k-1} \left(1 - \frac{1}{3(k-1)}\right)^{k-1}$$

for $k \ge 3$ ($\gamma_k > \beta_k$ for $k \ge 3$). Actually, they obtained a 0.7846-approximation algorithm by combining this algorithm with known MAX kSAT algorithms. They also proposed a generalization of this algorithm which finds a random truth assignment $\boldsymbol{x}^p$ with value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ at least

$$\begin{aligned} &0.914\hat{W}_1 + 0.750\hat{W}_2 + 0.750\hat{W}_3 + 0.766\hat{W}_4 \\ &+0.784\hat{W}_5 + 0.801\hat{W}_6 + \textstyle\sum_{k \ge 7} \gamma'_k \hat{W}_k, \end{aligned}$$

where

$$\gamma'_k = 1 - 0.914^k \left(1 - \frac{1}{k}\right)^k$$

for $k \ge 7$. They showed that if this is combined with Zwick's MAX SAT algorithm with conjectured 0.7977 performance guarantee then it leads to an approximation algorithm with performance guarantee 0.8331.

In this paper, we show that another generalization of the non-hybrid algorithms of Goemans and Williamson finds a random truth assignment $\boldsymbol{x}^p$ with value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ at least

$$\begin{aligned} &0.750\hat{W}_1 + 0.750\hat{W}_2 + 0.815\hat{W}_3 + 0.859\hat{W}_4 \\ &+0.894\hat{W}_5 + 0.920\hat{W}_6 + \textstyle\sum_{k \ge 7} \zeta_k \hat{W}_k, \end{aligned}$$

where

$$\zeta_k = 1 - \frac{1}{4}\left(\frac{3}{4}\right)^{k-2}$$

for $k \ge 3$ and $\zeta_k > \gamma_k$. We also present another algorithm which finds a random truth assignment $\boldsymbol{x}^p$ with value $F_{\mathcal{C}}(\boldsymbol{x}^p)$ at least

$$\begin{aligned} &0.914\hat{W}_1 + 0.750\hat{W}_2 + 0.757\hat{W}_3 + 0.774\hat{W}_4 \\ &+0.790\hat{W}_5 + 0.804\hat{W}_6 + \textstyle\sum_{k \ge 7} \gamma'_k \hat{W}_k. \end{aligned}$$

This will be used to obtain a 0.8353-approximation algorithm.

The remainder of the paper is structured as follows. In Section 3 we review the algorithms of Goemans and Williamson [3] and Asano and Williamson [1]. In Section 4 we give our main results and their proofs. In Section 5 we briefly outline improved approximation algorithms for MAX SAT obtained by our main results.

3 Goemans と Williamson のアルゴリズム

Goemans and Williamson considered the following LP relaxation (GW) of MAX SAT [3]:

$$\max \sum_{C_j \in \mathcal{C}} w(C_j) z_j$$

s.t.

$$\begin{aligned} \sum_{i=1}^{k_j} y_{j_i} \geq z_j & \quad \forall C_j = x_{j_1} \vee \cdots \vee x_{j_{k_j}} \in \mathcal{C} \\ y_i + y_{n+i} = 1 & \quad \forall i \in \{1,2,...,n\} \\ 0 \leq y_i \leq 1 & \quad \forall i \in \{1,2,...,2n\} \\ 0 \leq z_j \leq 1 & \quad \forall C_j \in \mathcal{C}. \end{aligned}$$

In this formulation, variables $\boldsymbol{y} = (y_i)$ correspond to the literals $\{x_1, \ldots, x_{2n}\}$ and variables $\boldsymbol{z} = (z_j)$ correspond to the clauses $\mathcal{C}$. Thus, variable $y_i = 1$ if and only if $x_i = 1$. Similarly, $z_j = 1$ if and only if C_j is satisfied. The first set of constraints implies that one of the literals in a clause must be true if the clause is satisfied and thus IP formulation of this (GW) with $y_i \in \{0,1\}$ $(\forall i \in \{1,2,...,2n\})$ and $z_j \in \{0,1\}$ $(\forall C_j \in \mathcal{C})$ exactly corresponds to MAX SAT.

Throughout this paper, let $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ be an optimal solution to this LP relaxation (GW) of MAX SAT. Goemans and Williamson set each variable x_i to be true with probability y_i^*. Then a clause $C_j = x_{j_1} \vee x_{j_2} \vee \cdots \vee x_{j_{k_j}}$ is satisfied by this random truth assignment $\boldsymbol{x}^p = \boldsymbol{y}^*$ with probability

$$C_j(\boldsymbol{y}^*) \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_j^*.$$

Thus, the expected value $F(\boldsymbol{y}^*)$ of $\boldsymbol{y}^*$ obtained in this way satisfies

$$\begin{aligned} F(\boldsymbol{y}^*) &= \sum_{C_j \in \mathcal{C}} w(C_j) C_j(\boldsymbol{y}^*) \\ &\geq \sum_{k \geq 1} \left(1 - \left(1 - \frac{1}{k}\right)^k\right) W_k^* \\ &\geq \left(1 - \frac{1}{e}\right) W^*, \end{aligned}$$

where $W^* = \sum_{C_j \in \mathcal{C}} w(C_j) z_j^*$ and $W_k^* = \sum_{C_j \in \mathcal{C}_k} w(C_j) z_j^*$. Since (GW) is an LP relaxation of MAX SAT, we have $W^* = \sum_{C_j \in \mathcal{C}} w(C_j) z_j^* \geq \hat{W} = \sum_{C_j \in \mathcal{C}} w(C_j) \hat{z}_j$ for an optimal solution $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ to the IP formulation of MAX SAT. Thus, this is a 0.632-approximation algorithm for MAX SAT, since $(1 - \frac{1}{e}) \approx 0.632$.

Goemans and Williamson [3] also considered three other non-linear randomized rounding algorithms. In these three algorithms, each variable x_i is set to be true with probability $f_\ell(y_i^*)$ defined as follows $(\ell = 1,2,3)$.

$$f_1(y) = \begin{cases} \frac{3}{4}y + \frac{1}{4} & \text{if } 0 \leq y \leq \frac{1}{3} \\ \frac{1}{2} & \text{if } \frac{1}{3} \leq y \leq \frac{2}{3} \\ \frac{3}{4}y & \text{if } \frac{2}{3} \leq y \leq 1, \end{cases}$$

$$f_2(y) = (2a-1)y + 1 - a \quad \left(\frac{3}{4} \leq a \leq \frac{3}{\sqrt[3]{4}} - 1\right),$$

$$1 - 4^{-y} \leq f_3(y) \leq 4^{y-1}.$$

Note that $f_\ell(y_i^*) + f_\ell(y_{n+i}^*) = 1$ hold for $\ell = 1,2$ and that $f_3(y_i^*)$ has to be chosen to satisfy $f_3(y_i^*) + f_3(y_{n+i}^*) = 1$. They then proved that all the random truth assignments $\boldsymbol{x}^p = f_\ell(\boldsymbol{y}^*) = (f_\ell(y_1^*), \ldots, f_\ell(y_{2n}^*))$ obtained in this way have the expected values at least $\frac{3}{4}W^*$ and lead to $\frac{3}{4}$-approximation algorithms. Asano and Williamson [1] sharpened the analysis of Goemans and Williamson to provide more precise bounds on the probability of a clause $C_j = x_{j_1} \vee x_{j_2} \vee \cdots \vee x_{j_k}$ with k literals being satisfied (and thus on the expected weight of satisfied clauses in $\mathcal{C}_k$) by the random truth assignment $\boldsymbol{x}^p = f_\ell(\boldsymbol{y}^*)$ for each k (and $\ell = 1,2$). From now on, we assume by symmetry, $x_{j_i} = x_i$ for each $i = 1,2,...,k$ since $f_\ell(x) = 1 - f_\ell(\bar{x})$ and we can set $x := \bar{x}$ if necessary. They considered clause $C_j = x_1 \vee x_2 \vee \cdots \vee x_k$ corresponding to the constraint $y_1 + y_2 + \cdots + y_k \geq z_j$ in the LP relaxation (GW) of MAX SAT, and gave a bound on the ratio of $C_j(f_\ell(\boldsymbol{y}^*))$ to z_j^*, where $C_j(f_\ell(\boldsymbol{y}^*))$ is the probability of clause C_j being satisfied by the random truth assignment $\boldsymbol{x}^p = f_\ell(\boldsymbol{y}^*)$ $(\ell = 1,2)$. Actually, they analyzed parametrized functions f_1^a and f_2^a with $\frac{1}{2} \leq a \leq 1$ defined as follows:

$$f_1^a(y) = \begin{cases} ay + 1 - a & \text{if } 0 \leq y \leq 1 - \frac{1}{2a} \\ \frac{1}{2} & \text{if } 1 - \frac{1}{2a} \leq y \leq \frac{1}{2a} \\ ay & \text{if } \frac{1}{2a} \leq y \leq 1, \end{cases}$$

$$f_2^a(y) = (2a-1)y + 1 - a.$$

Note that $f_1 = f_1^{3/4}$ and $f_2 = f_2^a$. Let

$$\gamma_{k,1}^a = 1 - \frac{1}{2}a^{k-1}\left(1 - \frac{1-\frac{1}{2a}}{k-1}\right)^{k-1},$$

$$\gamma_{k,2}^a = 1 - a^k\left(1 - \frac{1}{k}\right)^k,$$

$$\gamma_k^a = \begin{cases} a & \text{if } k=1 \\ \{\gamma_{k,1}^a, \gamma_{k,2}^a\} & \text{if } k \geq 2, \end{cases}$$

and

$$\delta_k^a = 1 - a^k\left(1 - \frac{2-\frac{1}{a}}{k}\right)^k.$$

Then their results are summarized as follows.

Proposition 3.1 [1] Let $\frac{1}{2} \leq a \leq 1$ and let $C_j(f_\ell^a(\boldsymbol{y}^*))$ be the probability of clause $C_j = x_1 \vee x_2 \vee \cdots \vee x_k \in \mathcal{C}$ satisfied by the random truth assignment $\boldsymbol{x}^p = f_\ell^a(\boldsymbol{y}^*)$ ($\ell = 1, 2$). Then $C_j(f_\ell^a(\boldsymbol{y}^*)) = 1 - \prod_{i=1}^k (1 - f_\ell^a(y_i^*))$ and the following statements hold.

1. $C_j(f_1^a(\boldsymbol{y}^*)) \geq \gamma_k^a z_j^*$ and the expected value $F(f_1^a(\boldsymbol{y}^*))$ satisfies $F(f_1^a(\boldsymbol{y}^*)) \geq \sum_{k\geq 1} \gamma_k^a W_k^*$.
2. $C_j(f_2^a(\boldsymbol{y}^*)) \geq \delta_k^a z_j^*$ and the expected value $F(f_2^a(\boldsymbol{y}^*))$ satisfies $F(f_2^a(\boldsymbol{y}^*)) \geq \sum_{k\geq 1} \delta_k^a W_k^*$.
3. $\gamma_k^a > \delta_k^a$ hold for all $k \geq 3$ and for all a with $\frac{1}{2} < a < 1$. $\gamma_k^a = \delta_k^a$ ($\gamma_1^a = \delta_1^a = a$, $\gamma_2^a = \delta_2^a = \frac{3}{4}$) hold for $k = 1, 2$.

4 主成果

Asano and Williamson did not consider a parametrized function of f_3. In this section we consider a parametrized function f_3^a of f_3 and show that it has better performance than f_1^a and f_2^a. Furthermore, its analysis (proof) is simpler. We also consider a generalization of both f_1^a and f_2^a.

For $\frac{1}{2} \leq a \leq 1$, let f_3^a be defined as follows:

$$f_3^a(y) = \begin{cases} 1 - \frac{a}{(4a^2)^y} & \text{if } 0 \leq y \leq \frac{1}{2} \\ \frac{(4a^2)^y}{4a} & \text{if } \frac{1}{2} \leq y \leq 1. \end{cases}$$

For $\frac{3}{4} \leq a \leq 1$, let

$$y_a = \frac{1}{a} - \frac{1}{2}.$$

Then the other parametrized function f_4^a is defined as follows:

$$f_4^a(y) = \begin{cases} ay + 1 - a & \text{if } 0 \leq y \leq 1 - y_a \\ \frac{a}{2}y + \frac{1}{2} - \frac{a}{4} & \text{if } 1 - y_a \leq y \leq y_a \\ ay & \text{if } y_a \leq y \leq 1. \end{cases}$$

Thus, $f_3^a(y) + f_3^a(1-y) = 1$, $f_4^a(y) + f_4^a(1-y) = 1$ hold for $0 \leq y \leq 1$. Furthermore, f_3^a and f_4^a are both continuous functions which are increasing with y. Thus, $f_3^a(\frac{1}{2}) = f_4^a(\frac{1}{2}) = \frac{1}{2}$. Let ζ_k^a and η_k^a be the numbers defined as follows.

$$\zeta_k^a = \begin{cases} a & \text{if } k=1 \\ 1 - \frac{1}{4}a^{k-2} & \text{if } k \geq 2, \end{cases}$$

$$\eta_{k,1}^a = 1 - a^k\left(1 - \frac{1}{k}\right)^k,$$

$$\eta_{k,2}^a = 1 - \frac{a^{k-2}}{4},$$

$$\eta_{k,3}^a = 1 - \frac{a^k}{2}\left(1 - \frac{1-y_a}{k-1}\right)^{k-1},$$

$$\eta_{k,4}^a = 1 - \frac{1}{2^k}\left(1 + \frac{a}{2} - \frac{a}{k}\right)^k,$$

$$\eta_k^a = \begin{cases} a & \text{if } k=1 \\ \min\{\eta_{k,1}^a, \eta_{k,2}^a, \eta_{k,3}^a, \eta_{k,4}^a\} & \text{if } k \geq 2 \end{cases}$$

($\eta_{k,1}^a = \gamma_{k,2}^a$ and $\eta_{k,2}^a = \zeta_k^a$). Then we have the following theorems for the two parameterized functions f_3^a and f_4^a.

Theorem 4.1 For $\frac{1}{2} \leq a \leq \frac{\sqrt{e}}{2} = 0.82436$, the probability of $C_j = x_1 \vee x_2 \vee \cdots \vee x_k \in \mathcal{C}$ being satisfied by the random truth assignment $\boldsymbol{x}^p = f_3^a(\boldsymbol{y}^*) = (f_3^a(y_1^*), \ldots, f_3^a(y_{2n}^*))$ is

$$C_j(f_3^a(\boldsymbol{y}^*)) = 1 - \prod_{i=1}^k (1 - f_3^a(y_i^*)) \geq \zeta_k^a z_j^*.$$

Thus, the expected value $F(f_3^a(\boldsymbol{y}^*))$ of $\boldsymbol{x}^p = f_3^a(\boldsymbol{y}^*)$ satisfies $F(f_3^a(\boldsymbol{y}^*)) \geq \sum_{k\geq 1} \zeta_k^a W_k^*$.

Theorem 4.2 For $\frac{\sqrt{e}}{2} = 0.82436 \leq a \leq 1$, the probability of $C_j = x_1 \vee x_2 \vee \cdots \vee x_k \in \mathcal{C}$ being satisfied by the random truth assignment $\boldsymbol{x}^p = f_4^a(\boldsymbol{y}^*) = (f_4^a(y_1^*), \ldots, f_4^a(y_{2n}^*))$ is

$$C_j(f_4^a(\boldsymbol{y}^*)) = 1 - \prod_{i=1}^k (1 - f_4^a(y_i^*)) \geq \eta_k^a z_j^*.$$

Thus, the expected value $F(f_4^a(\boldsymbol{y}^*))$ of $\boldsymbol{x}^p = f_4^a(\boldsymbol{y}^*)$ satisfies $F(f_4^a(\boldsymbol{y}^*)) \geq \sum_{k\geq 1} \eta_k^a W_k^*$.

Theorem 4.3 The following statements hold for γ_k^a, δ_k^a, ζ_k^a, and η_k^a.

1. If $\frac{1}{2} \leq a \leq \frac{\sqrt{e}}{2} = 0.82436$, then $\zeta_k^a > \gamma_k^a > \delta_k^a$ hold for all $k \geq 3$.
2. If $\frac{\sqrt{e}}{2} = 0.82436 \leq a < 1$, then $\eta_k^a \geq \gamma_k^a > \delta_k^a$ hold for all $k \geq 3$. In particular, if $\frac{\sqrt{e}}{2} = 0.82436 \leq a \leq 0.881611$, then $\eta_k^a > \gamma_k^a > \delta_k^a$ hold for all $k \geq 3$.
3. For $k = 1, 2$, $\gamma_k^a = \delta_k^a = \zeta_k^a$ hold if $\frac{1}{2} \leq a \leq \frac{\sqrt{e}}{2} = 0.82436$, and $\gamma_k^a = \delta_k^a = \eta_k^a$ hold if $\frac{\sqrt{e}}{2} = 0.82436 \leq a \leq 1$.

5 改善アルゴリズム

In this section, we briefly outline our improved approximation algorithms for MAX SAT based on a hybrid approach which is described in detail in Asano and Williamson [1]. We use a semidefinite programming relaxation of MAX SAT which is a combination of ones given by Goemans and Williamson [4], Feige and Goemans [2], Karloff and Zwick [8], Halperin and Zwick [6], and Zwick [9]. Our algorithms pick the best solution returned by the four algorithms corresponding to (1) f_3^a in Goemans and Williamson [3], (2) MAX 2SAT algorithm of Feige and Goemans [2] or of Halperin and Zwick [6], (3) MAX 3SAT algorithm of Karloff and Zwick [8] or of Halperin and Zwick [6], and (4) Zwick's MAX SAT algorithm with a conjectured performance guarantee 0.7977 [9]. The expected value of the solution is at least as good as the expected value of an algorithm that uses Algorithm (i) with probability p_i, where $p_1 + p_2 + p_3 + p_4 = 1$.

Our first algorithm pick the best solution returned by the three algorithms corresponding to (1) f_3^a in Goemans and Williamson [3], (2) Feige and Goemans's MAX 2SAT algorithm [2], and (3) Karloff and Zwick's MAX 3SAT algorithm [8] (this implies that $p_4 = 0$). From the arguments in Section 3, the probability that a clause $C_j \in \mathcal{C}_k$ is satisfied by Algorithm (1) is at least $\zeta_k^a z_j^*$, where ζ_k^a is defined in Eq.(1). Similarly, from the arguments in [4, 2], the probability that a clause $C_j \in \mathcal{C}_k$ is satisfied by Algorithm (2) is

$$\text{at least} \quad 0.93109 \cdot \frac{2}{k} z_j^* \quad \text{for } k \geq 2,$$
$$\text{and at least} \quad 0.97653 z_j^* \quad \text{for } k = 1.$$

By an analysis obtained by Karloff and Zwick [8] and an argument similar to one in [4], the probability that a clause $C_j \in \mathcal{C}_k$ is satisfied by Algorithm (3) is at least

$$\text{at least} \quad \frac{3}{k}\frac{7}{8} z_j^* \quad \text{for } k \geq 3,$$
$$\text{and at least} \quad 0.87856 z_j^* \quad \text{for } k = 1, 2.$$

Suppose that we set $a = 0.74054$, $p_1 = 0.7861$, $p_2 = 0.1637$, and $p_3 = 0.0502$ ($p_4 = 0$). Then

$$a p_1 + 0.97653 p_2 + 0.87856 p_3 \geq 0.7860 \quad \text{for } k = 1,$$
$$\frac{3}{4} p_1 + 0.93109 p_2 + 0.87856 p_3 \geq 0.7860 \quad \text{for } k = 2,$$
$$\zeta_k^a p_1 + \frac{2 \times 0.93109}{k} p_2 + \frac{3}{k}\frac{7}{8} p_3 \geq 0.7860 \quad \text{for } k \geq 3.$$

Thus this is a 0.7860-approximation algorithm. Note that, under same conditions in Asano and Williamson [1], the algorithm picking the best solution returned by the three algorithms corresponding to (1) f_1^a with $a = \frac{3}{4}$ in Goemans and Williamson [3], (2) Feige and Goemans [2], and (3) Karloff and Zwick [8] only achieves the performance guarantee 0.7846.

Suppose next that we use three algorithms (1) f_3^a in Goemans and Williamson [3], (2) Halperin and Zwick's MAX 2SAT algorithm [6], and (3) Halperin and Zwick's MAX 3SAT algorithm [6] instead of Feige and Goemans [2] and Karloff and Zwick [8]. If we set $a = 0.739634$, $p_1 = 0.787777$, $p_2 = 0.157346$, and $p_3 = 0.054877$, then we have

$$a p_1 + 0.9828 p_2 + 0.9197 p_3 \geq 0.7877 \quad \text{for } k = 1,$$
$$\frac{3}{4} p_1 + 0.9309 p_2 + 0.9197 p_3 \geq 0.7877 \quad \text{for } k = 2,$$
$$\zeta_k^a p_1 + \frac{2 \times 0.9309}{k} p_2 + \frac{3}{k}\frac{7}{8} p_3 \geq 0.7877 \quad \text{for } k \geq 3.$$

Thus we have a 0.7877-approximation algorithm for MAX SAT (note that the performance guarantees of Halperin and Zwick's MAX 2SAT and MAX 3SAT algorithms are based on the numerical evidence [6]).

Suppose finally that we use two algorithms (1) f_4^a in Goemans and Williamson [3] and (4) Zwick's MAX

SAT algorithm with a conjectured performance guarantee 0.7977 [9]. If we set $a = 0.907180$, $p_1 = 0.343137$ and $p_4 = 0.656863$ ($p_2 = p_3 = 0$), then the probability of clause C_j with k literals being satisfied can be shown to be at least $0.8353z_j^*$ for each $k \geq 1$. Thus, we can obtain a 0.8353-approximation algorithm for MAX SAT if a conjectured performance guarantee 0.7977 is true in Zwick's MAX SAT algorithm [9].

参 考 文 献

[1] T. Asano and D.P. Williamson, Improved approximation algorithms for MAX SAT, *Journal of Algorithms* 42, pp.173–202, 2002.

[2] U. Feige and M. X. Goemans, Approximating the value of two prover proof systems, with applications to MAX 2SAT and MAX DICUT, In *Proc. 3rd Israel Symposium on Theory of Computing and Systems*, pp. 182–189, 1995.

[3] M. X. Goemans and D. P. Williamson, New 3/4-approximation algorithms for the maximum satisfiability problem, *SIAM Journal on Discrete Mathematics* 7, pp. 656–666, 1994.

[4] M. X. Goemans and D. P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, *Journal of the ACM* 42, pp. 1115–1145, 1995.

[5] J. Håstad, Some optimal inapproximability results, In *Proc. 28th ACM Symposium on the Theory of Computing*, pp. 1–10, 1997.

[6] E. Halperin and U. Zwick, Approximation algorithms for MAX 4-SAT and rounding procedures for semidefinite programs, *Journal of Algorithms* 40, pp.184–211, 2001.

[7] D. S. Johnson, Approximation algorithms for combinatorial problems, *Journal of Computer and Systems Science* 9, pp. 256–278, 1974.

[8] H. Karloff and U. Zwick, A 7/8-approximation algorithm for MAX 3SAT?, In *Proc. 38th IEEE Symposium on the Foundations of Computer Science*, pp. 406–415, 1997.

[9] U. Zwick, Outward rotations: a tool for rounding solutions of semidefinite programming relaxations, with applications to MAX CUT and other problems, In *Proc. 31st ACM Symposium on the Theory of Computing*, pp. 679–687, 1999.